

A STOCHASTIC INTEGRAL OF OPERATOR-VALUED FUNCTIONS

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To Professor M. L. Gorbachuk on the occasion of his 70th birthday.

ABSTRACT. In this note we define and study a Hilbert space-valued stochastic integral of operator-valued functions with respect to Hilbert space-valued measures. We show that this integral generalizes the classical Itô stochastic integral of adapted processes with respect to normal martingales and the Itô integral in a Fock space.

1. INTRODUCTION

Here and subsequently, we fix a real number $T > 0$. Let \mathcal{H} be a complex Hilbert space, M be a fixed vector from \mathcal{H} and $[0, T] \ni t \mapsto E_t$ be a resolution of identity in \mathcal{H} . Consider the \mathcal{H} -valued function (*abstract martingale*)

$$[0, T] \ni t \mapsto M_t := E_t M \in \mathcal{H}.$$

In this paper we construct and study an integral

$$(1) \quad \int_{[0, T]} A(t) dM_t$$

for a certain class of operator-valued functions $[0, T] \ni t \mapsto A(t)$ whose values are linear operators in the space \mathcal{H} . We define such an integral as an element of the Hilbert space \mathcal{H} and call it a *Hilbert space-valued stochastic integral* (or *H-stochastic integral*). By analogy with the classical integration theory we first define integral (1) for a certain class of simple operator-valued functions and then extend this definition to a wider class.

We illustrate our abstract constructions with a few examples. Thus, we show that the classical Itô stochastic integral is a particular case of the *H-stochastic integral*. Namely, let $\mathcal{H} := L^2(\Omega, \mathcal{A}, P)$ be a space of square integrable functions on a complete probability space (Ω, \mathcal{A}, P) , $\{\mathcal{A}_t\}_{t \in [0, T]}$ be a filtration satisfying the usual conditions and $\{N_t\}_{t \in [0, T]}$ be a normal martingale on (Ω, \mathcal{A}, P) with respect to $\{\mathcal{A}_t\}_{t \in [0, T]}$, i.e.,

$$\{N_t\}_{t \in [0, T]} \quad \text{and} \quad \{N_t^2 - t\}_{t \in [0, T]}$$

are martingales for $\{\mathcal{A}_t\}_{t \in [0, T]}$. It follows from the properties of martingales that

$$N_t = \mathbb{E}[N_T | \mathcal{A}_t], \quad t \in [0, T],$$

where $\mathbb{E}[\cdot | \mathcal{A}_t]$ is a conditional expectation with respect to the σ -algebra \mathcal{A}_t . It is well known that $\mathbb{E}[\cdot | \mathcal{A}_t]$ is the orthogonal projector in the space $L^2(\Omega, \mathcal{A}, P)$ onto its subspace $L^2(\Omega, \mathcal{A}_t, P)$ and, moreover, the corresponding projector-valued function $\mathbb{R}_+ \ni t \mapsto E_t := \mathbb{E}[\cdot | \mathcal{A}_t]$ is a resolution of identity in $L^2(\Omega, \mathcal{A}, P)$, see e.g. [13, 3, 4, 12, 7]. In this way the normal martingale $\{N_t\}_{t \in [0, T]}$ can be interpreted as an abstract martingale, i.e.,

$$[0, T] \ni t \mapsto N_t = \mathbb{E}[N_T | \mathcal{A}_t] = E_t N_T \in \mathcal{H}.$$

Hence, in the space $L^2(\Omega, \mathcal{A}, P)$ we can construct the *H-stochastic integral* with respect to the normal martingale N_t . Let $F \in L^2([0, T] \times \Omega, dt \times P)$ be a square integrable

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stochastic process adapted to the filtration $\{\mathcal{A}_t\}_{t \in [0, T]}$. We consider the operator-valued function $[0, T] \ni t \mapsto A_F(t)$ whose values are operators $A_F(t)$ of multiplication by the function $F(t) = F(t, \cdot) \in L^2(\Omega, \mathcal{A}, P)$ in the space $L^2(\Omega, \mathcal{A}, P)$,

$$L^2(\Omega, \mathcal{A}, P) \supset \text{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, \mathcal{A}, P).$$

In this paper we prove that the H -stochastic integral of $[0, T] \ni t \mapsto A_F(t)$ coincides with the classical Itô stochastic integral $\int_{[0, T]} F(t) dN_t$ of F . That is,

$$\int_{[0, T]} A_F(t) dN_t = \int_{[0, T]} F(t) dN_t.$$

In the last part of this note we show that the Itô integral in a Fock space is the H -stochastic integral and establish a connection of such an integral with the classical Itô stochastic integral. The corresponding results are given without proofs (the proofs will be given in a forthcoming publication). Note that the Itô integral in a Fock space is a useful tool in the quantum stochastic calculus, see e.g. [2] for more details.

We remark that in [3, 4] the authors gave a definition of the operator-valued stochastic integral

$$B := \int_{[0, T]} A(t) dE_t$$

for a family $\{A(t)\}_{t \in [0, T]}$ of commuting normal operators in \mathcal{H} . Such an integral was defined using a spectral theory of commuting normal operators. It is clear that for a fixed vector $M \in \text{Dom}(B) \subset \mathcal{H}$ the formula

$$\int_{[0, T]} A(t) dM_t := \left(\int_{[0, T]} A(t) dE_t \right) M$$

can be regarded as a definition of integral (1). In this way we obtain another definition of integral (1) different from the one we have proposed in this paper.

2. THE CONSTRUCTION OF THE H -STOCHASTIC INTEGRAL

Let \mathcal{H} be a complex Hilbert space, $\mathcal{L}(\mathcal{H})$ be a space of all bounded linear operators in \mathcal{H} , $M \neq 0$ be a fixed vector from \mathcal{H} and

$$[0, T] \ni t \mapsto E_t \in \mathcal{L}(\mathcal{H})$$

be a resolution of identity in \mathcal{H} , that is a right-continuous increasing family of orthogonal projections in \mathcal{H} such that $E_T = 1$. Note that the resolution of identity E can be regarded as a projector-valued measure $\mathcal{B}([0, T]) \ni \alpha \mapsto E(\alpha) \in \mathcal{L}(\mathcal{H})$ on the Borel σ -algebra $\mathcal{B}([0, T])$. Namely, for any interval $(s, t] \subset [0, T]$ we set

$$E((s, t]) := E_t - E_s, \quad E(\{0\}) := E_0, \quad E(\emptyset) := 0,$$

and extend this definition to all Borel subsets of $[0, T]$, see e.g. [6] for more details.

By definition, the \mathcal{H} -valued function

$$[0, T] \ni t \mapsto M_t := E_t M \in \mathcal{H}$$

is an *abstract martingale* in the Hilbert space \mathcal{H} .

In this section we give a definition of integral (1) for a certain class of operator-valued functions with respect to the abstract martingale M_t . A construction of such an integral is given step-by-step, beginning with the simplest class of operator-valued functions. Let us introduce the required class of simple functions.

For each point $t \in [0, T]$, we denote by

$$\mathcal{H}_M(t) := \text{span}\{M_{s_2} - M_{s_1} \mid (s_1, s_2] \subset (t, T]\} \subset \mathcal{H}$$

the linear span of the set $\{M_{s_2} - M_{s_1} \mid (s_1, s_2] \subset (t, T]\}$ in \mathcal{H} and by

$$\mathcal{L}_M(t) = \mathcal{L}(\mathcal{H}_M(t) \rightarrow \mathcal{H})$$

the set of all linear operators in \mathcal{H} that continuously act from $\mathcal{H}_M(t)$ to \mathcal{H} . The increasing family $\mathcal{L}_M = \{\mathcal{L}_M(t)\}_{t \in [0, T]}$ will play here a role of the filtration $\{\mathcal{A}_t\}_{t \in [0, T]}$ in the classical martingale theory.

For a fixed $t \in [0, T]$, a linear operator A in \mathcal{H} will be called $\mathcal{L}_M(t)$ -measurable if

(i) $A \in \mathcal{L}_M(t)$ and, for all $s \in [t, T]$,

$$\|A\|_{\mathcal{L}_M(t)} = \|A\|_{\mathcal{L}_M(s)} := \sup \left\{ \frac{\|Ag\|_{\mathcal{H}}}{\|g\|_{\mathcal{H}}} \mid g \in \mathcal{H}_M(s), g \neq 0 \right\}.$$

(ii) A is partially commuting with the resolution of identity E . More precisely,

$$AE_s g = E_s A g, \quad g \in \mathcal{H}_M(t), \quad s \in [t, T].$$

Such a definition of $\mathcal{L}_M(t)$ -measurability is motivated by a number of reasons:

- $\mathcal{L}_M(t)$ -measurability is a natural generalization of the usual \mathcal{A}_t -measurability in classical stochastic calculus, see Lemma 1 (Section 3) for more details;
- in some sense, $\mathcal{L}_M(t)$ -measurability (for each t) is the minimal restriction on the behavior of a simple operator-valued function $[0, T] \ni t \mapsto A(t)$ that will allow us to obtain an analogue of the Itô isometry property (see inequality (4) below) and to extend the H -stochastic integral from a simple class of functions to a wider one.

In what follows, it is convenient for us to call $\mathcal{L}_M(T)$ -measurable all linear operators in \mathcal{H} . Evidently, if a linear operator A in \mathcal{H} is $\mathcal{L}_M(t)$ -measurable for some $t \in [0, T]$ then A is $\mathcal{L}_M(s)$ -measurable for all $s \in [t, T]$.

A family $\{A(t)\}_{t \in [0, T]}$ of linear operators in \mathcal{H} will be called a *simple \mathcal{L}_M -adapted operator-valued function on $[0, T]$* if, for each $t \in [0, T]$, the operator $A(t)$ is $\mathcal{L}_M(t)$ -measurable and there exists a partition $0 = t_0 < t_1 < \dots < t_n = T$ of $[0, T]$ such that

$$(2) \quad A(t) = \sum_{k=0}^{n-1} A_k \chi_{(t_k, t_{k+1}]}(t), \quad t \in [0, T],$$

where $\chi_\alpha(\cdot)$ is the characteristic function of the Borel set $\alpha \in \mathcal{B}([0, T])$.

Let $S = S(M)$ denote the space of all simple \mathcal{L}_M -adapted operator-valued functions on $[0, T]$. For a function $A \in S$ with representation (2) we define an H -stochastic integral of A with respect to the abstract martingale M_t through the formula

$$(3) \quad \int_{[0, T]} A(t) dM_t := \sum_{k=0}^{n-1} A_k (M_{t_{k+1}} - M_{t_k}) \in \mathcal{H}.$$

We can show that this definition does not depend on the choice of representation of the simple function A in the space S .

In the space S we introduce a quasinorm by setting

$$\|A\|_{S_2} := \left(\int_{[0, T]} \|A(t)\|_{\mathcal{L}_M(t)}^2 d\mu(t) \right)^{\frac{1}{2}} := \left(\sum_{k=0}^{n-1} \|A_k\|_{\mathcal{L}_M(t_k)}^2 \mu((t_k, t_{k+1}]) \right)^{\frac{1}{2}}$$

for each $A \in S$ with representation (2). Here the measure μ is defined by the formula

$$\mathcal{B}([0, T]) \ni \alpha \mapsto \mu(\alpha) := \|M(\alpha)\|_{\mathcal{H}}^2 = (E(\alpha)M, M)_{\mathcal{H}} \in \mathbb{R}_+,$$

where $M(\alpha) := E(\alpha)M$ for all $\alpha \in \mathcal{B}([0, T])$, in particular,

$$M((t_k, t_{k+1}]) := E((t_k, t_{k+1}])M = M_{t_{k+1}} - M_{t_k}, \quad (t_k, t_{k+1}] \subset [0, T].$$

The following statement is fundamental.

Theorem 1. *Let $A, B \in S$ and $a, b \in \mathbb{C}$. Then*

$$\int_{[0, T]} (aA(t) + bB(t)) dM_t = a \int_{[0, T]} A(t) dM_t + b \int_{[0, T]} B(t) dM_t$$

and

$$(4) \quad \left\| \int_{[0,T]} A(t) dM_t \right\|_{\mathcal{H}}^2 \leq \int_{[0,T]} \|A(t)\|_{\mathcal{L}_M(t)}^2 d\mu(t).$$

Proof. The first assertion is trivial.

Let us check inequality (4). Using (i), (ii) and properties of the resolution of identity E , for $A \in S$ with representation (2), we obtain

$$\begin{aligned} \left\| \int_{[0,T]} A(t) dM_t \right\|_{\mathcal{H}}^2 &= \left(\int_{[0,T]} A(t) dM_t, \int_{[0,T]} A(t) dM_t \right)_{\mathcal{H}} \\ &= \sum_{k,m=0}^{n-1} (A_k M(\Delta_k), A_m M(\Delta_m))_{\mathcal{H}} \\ &= \sum_{k,m=0}^{n-1} (A_k E(\Delta_k) M, A_m E(\Delta_m) M)_{\mathcal{H}} \\ &= \sum_{k,m=0}^{n-1} (E(\Delta_k) A_k E(\Delta_k) M, E(\Delta_m) A_m E(\Delta_m) M)_{\mathcal{H}} \\ &= \sum_{k=0}^{n-1} (A_k E(\Delta_k) M, A_k E(\Delta_k) M)_{\mathcal{H}} = \sum_{k=0}^{n-1} \|A_k M(\Delta_k)\|_{\mathcal{H}}^2 \\ &\leq \sum_{k=0}^{n-1} \|A_k\|_{\mathcal{L}_M(t_k)}^2 \|M(\Delta_k)\|_{\mathcal{H}}^2 = \sum_{k=0}^{n-1} \|A_k\|_{\mathcal{L}_M(t_k)}^2 \mu(\Delta_k) \\ &= \int_{[0,T]} \|A(t)\|_{\mathcal{L}_M(t)}^2 d\mu(t), \end{aligned}$$

where $\Delta_k := (t_k, t_{k+1}]$ for all $k \in \{0, \dots, n-1\}$. \square

Inequality (4) enables us to extend the H -stochastic integral to operator-valued functions $[0, T] \ni t \mapsto A(t)$ which are not necessarily simple. Namely, denote by $S_2 = S_2(M)$ a Banach space associated with the quasinorm $\|\cdot\|_{S_2}$. For its construction, it is first necessary to pass from S to the factor space

$$\dot{S} := S / \{A \in S \mid \|A\|_{S_2} = 0\}$$

and then to take the completion of \dot{S} . It is not difficult to see that elements of the space S_2 are equivalence classes of operator-valued functions on $[0, T]$ whose values are linear operators in the space \mathcal{H} .

An operator-valued function $[0, T] \ni t \mapsto A(t)$ will be called *H -stochastic integrable* with respect to M_t if A belongs to the space S_2 . It follows from the definition of the space S_2 that for each $A \in S_2$ there exists a sequence $(A_n)_{n=0}^{\infty}$ of simple operator-valued functions $A_n \in S$ such that

$$(5) \quad \int_{[0,T]} \|A(t) - A_n(t)\|_{\mathcal{L}_M(t)}^2 d\mu(t) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Due to (4), for such a sequence $(A_n)_{n=0}^{\infty}$, the limit

$$\lim_{n \rightarrow \infty} \int_{[0,T]} A_n(t) dM_t$$

exists in \mathcal{H} and does not depend on the choice of the sequence $(A_n)_{n=0}^{\infty} \subset S$ satisfying (5). We denote this limit by

$$\int_{[0,T]} A(t) dM_t := \lim_{n \rightarrow \infty} \int_{[0,T]} A_n(t) dM_t$$

and call it the *H -stochastic integral of $A \in S_2$* with respect to the abstract martingale M_t . It is clear that for all $A \in S_2$ the assertions of Theorem 1 still hold.

Note one simple property of the integral introduced above. Let U be some unitary operator acting from \mathcal{H} onto another complex Hilbert space \mathcal{K} . Then

$$[0, T] \ni t \mapsto G_t := UM_t \in \mathcal{K}$$

is an abstract martingale in the space \mathcal{K} because, for any $t \in [0, T]$,

$$G_t = UM_t = X_t G, \quad X_t := UE_t U^{-1}, \quad G := UM \in \mathcal{K},$$

and X_t is a resolution of identity in the space \mathcal{K} .

Let an operator-valued function $[0, T] \ni t \mapsto A(t)$ be H -stochastic integrable with respect to M_t . One can show that the operator-valued function $[0, T] \ni t \mapsto UA(t)U^{-1}$ is H -stochastic integrable with respect to G_t and

$$U \left(\int_{[0, T]} A(t) dM_t \right) = \int_{[0, T]} UA(t)U^{-1} dG_t \in \mathcal{K}.$$

3. THE ITÔ STOCHASTIC INTEGRAL AS AN H -STOCHASTIC INTEGRAL

Let (Ω, \mathcal{A}, P) be a complete probability space and $\{\mathcal{A}_t\}_{t \in [0, T]}$ be a right continuous filtration. Suppose that the σ -algebra \mathcal{A}_0 contains all P -null sets of \mathcal{A} and $\mathcal{A} = \mathcal{A}_T$. Moreover, we assume that \mathcal{A}_0 is trivial, i.e., every set $\alpha \in \mathcal{A}_0$ has probability 0 or 1.

Let $N = \{N_t\}_{t \in [0, T]}$ be a *normal martingale* on (Ω, \mathcal{A}, P) with respect to $\{\mathcal{A}_t\}_{t \in [0, T]}$. That is, $N_t \in L^2(\Omega, \mathcal{A}_t, P)$ for all $t \in [0, T]$ and

$$\mathbb{E}[N_t - N_s | \mathcal{A}_s] = 0, \quad \mathbb{E}[(N_t - N_s)^2 | \mathcal{A}_s] = t - s$$

for all $s, t \in [0, T]$ such that $s < t$. Without loss of generality one can assume that $N_0 = 0$. Note that there are many examples of normal martingales, — the Brownian motion, the compensated Poisson process, the Azéma martingales and others, see for instance [10, 8, 12].

We will denote by $L_a^2([0, T] \times \Omega)$ the set of all functions (equivalence classes), adapted to the filtration $\{\mathcal{A}_t\}_{t \in [0, T]}$, from the space

$$L^2([0, T] \times \Omega) := L^2([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{A}, dt \times P)$$

where dt is the Lebesgue measure on $\mathcal{B}([0, T])$.

Let us show that the *Itô stochastic integral* $\int_{[0, T]} F(t) dN_t$ of $F \in L_a^2([0, T] \times \Omega)$ with respect to the normal martingale N can be considered as an H -stochastic integral (see e.g. [15, 16] for the definition and properties of the classical Itô integral). To this end, we set $\mathcal{H} := L^2(\Omega, \mathcal{A}, P)$ and consider, in this space, the resolution of identity

$$[0, T] \ni t \mapsto E_t := \mathbb{E}[\cdot | \mathcal{A}_t] \in \mathcal{L}(\mathcal{H})$$

generated by the filtration $\{\mathcal{A}_t\}_{t \in [0, T]}$. Let $M := N_T \in L^2(\Omega, \mathcal{A}, P)$, then the corresponding abstract martingale

$$[0, T] \ni t \mapsto N_t := E_t N_T = \mathbb{E}[N_T | \mathcal{A}_t] \in \mathcal{H}$$

is our normal martingale. Note also that

$$\mu([0, t]) = \|N([0, t])\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \|N_t\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \mathbb{E}[N_t^2] = \mathbb{E}[N_t^2 | \mathcal{A}_0] = t,$$

i.e., μ is the Lebesgue measure on $\mathcal{B}([0, T])$.

In the context of this section, $\mathcal{L}_M(t)$ -measurability is equivalent to the usual \mathcal{A}_t -measurability. More precisely, the following result holds.

Lemma 1. *Let $t \in [0, T]$. For given $F \in L^2(\Omega, \mathcal{A}, P)$ the operator A_F of multiplication by the function F in the space $L^2(\Omega, \mathcal{A}, P)$ is $\mathcal{L}_N(t)$ -measurable if and only if the function F is \mathcal{A}_t -measurable, i.e., $F = \mathbb{E}[F | \mathcal{A}_t]$. Moreover, if $F \in L^2(\Omega, \mathcal{A}, P)$ is an \mathcal{A}_t -measurable function then*

$$(6) \quad \|A_F\|_{\mathcal{L}_N(t)} = \|A_F\|_{\mathcal{L}_N(s)} = \|F\|_{L^2(\Omega, \mathcal{A}, P)}, \quad s \in [t, T].$$

Proof. Suppose $F \in L^2(\Omega, \mathcal{A}, P)$ is an \mathcal{A}_t -measurable function. Let us show that the operator A_F is $\mathcal{L}_N(t)$ -measurable.

First, we prove that $A_F \in \mathcal{L}_N(t)$. Taking into account that F is an \mathcal{A}_t -measurable function, $\{N_t\}_{t \in [0, T]}$ is the normal martingale and the σ -algebra \mathcal{A}_0 is trivial, for any interval $(s_1, s_2] \subset (t, T]$, we obtain

$$\begin{aligned} \|A_F(N_{s_2} - N_{s_1})\|_{L^2(\Omega, \mathcal{A}, P)}^2 &= \|F(N_{s_2} - N_{s_1})\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \mathbb{E}[F^2(N_{s_2} - N_{s_1})^2] \\ &= \mathbb{E}[F^2(N_{s_2} - N_{s_1})^2 | \mathcal{A}_0] = \mathbb{E}[F^2 \mathbb{E}[(N_{s_2} - N_{s_1})^2 | \mathcal{A}_{s_1}] | \mathcal{A}_0] \\ &= \mathbb{E}[F^2 \mathbb{E}[(N_{s_2} - N_{s_1})^2 | \mathcal{A}_{s_1}]] = \mathbb{E}[F^2](s_2 - s_1) \\ &= \mathbb{E}[F^2] \mathbb{E}[(N_{s_2} - N_{s_1})^2] \\ &= \|F\|_{L^2(\Omega, \mathcal{A}, P)}^2 \|N_{s_2} - N_{s_1}\|_{L^2(\Omega, \mathcal{A}, P)}^2. \end{aligned}$$

We can similarly show that

$$\|A_F G\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \|F\|_{L^2(\Omega, \mathcal{A}, P)}^2 \|G\|_{L^2(\Omega, \mathcal{A}, P)}^2$$

for all $G \in \mathcal{H}_N(t) = \text{span}\{N_{s_2} - N_{s_1} \mid (s_1, s_2] \subset (t, T]\}$. Hence $A_F \in \mathcal{L}_N(t)$ and, moreover, equality (6) takes place.

Let us check that A_F is partially commuting with E , i.e.,

$$A_F E_s G = E_s A_F G, \quad G \in \mathcal{H}_N(t), \quad s \in [t, T].$$

Since $F \in L^2(\Omega, \mathcal{A}, P)$ is an \mathcal{A}_t -measurable function and $FG \in L^2(\Omega, \mathcal{A}, P)$, for any $s \in [t, T]$ and any function $G \in \mathcal{H}_N(t)$, we have

$$A_F E_s G = F E_s G = F \mathbb{E}[G | \mathcal{A}_s] = \mathbb{E}[FG | \mathcal{A}_s] = E_s A_F G.$$

Thus, the first part of the lemma is proved.

Let us prove the converse statement of the lemma: if for a given $F \in L^2(\Omega, \mathcal{A}, P)$ the operator A_F is $\mathcal{L}_N(t)$ -measurable then F is an \mathcal{A}_t -measurable function.

Since A_F is an $\mathcal{L}_N(t)$ -measurable operator, we see that for any $s \in [t, T]$

$$A_F E_s G = E_s A_F G, \quad G \in \mathcal{H}_N(t),$$

or, equivalently,

$$(7) \quad A_F \mathbb{E}[G | \mathcal{A}_s] = \mathbb{E}[A_F G | \mathcal{A}_s], \quad G \in \mathcal{H}_N(t).$$

Let $s \in (t, T)$ and $(s_1, s_2] \subset (t, s]$. We take

$$G := N_{s_2} - N_{s_1} \in \mathcal{H}_N(t).$$

Evidently, G is an \mathcal{A}_s -measurable function and

$$A_F \mathbb{E}[G | \mathcal{A}_s] = A_F G = FG, \quad \mathbb{E}[A_F G | \mathcal{A}_s] = \mathbb{E}[FG | \mathcal{A}_s] = G \mathbb{E}[F | \mathcal{A}_s].$$

Hence, using (7), we obtain

$$FG = G \mathbb{E}[F | \mathcal{A}_s].$$

As a result,

$$F = \mathbb{E}[F | \mathcal{A}_s], \quad s \in (t, T].$$

Since the resolution of identity $[0, T] \ni s \mapsto E_s = \mathbb{E}[\cdot | \mathcal{A}_s] \in \mathcal{L}(\mathcal{H})$ is a right-continuous function, the latter equality still holds for $s = t$, and therefore F is an \mathcal{A}_t -measurable function. \square

As a simple consequence of Lemma 1 we obtain the following result.

Theorem 2. *Let F belong to $L^2([0, T] \times \Omega)$. The family $\{A_F(t)\}_{t \in [0, T]}$ of the operators $A_F(t)$ of multiplication by $F(t) = F(t, \cdot) \in L^2(\Omega, \mathcal{A}, P)$ in the space $L^2(\Omega, \mathcal{A}, P)$,*

$$L^2(\Omega, \mathcal{A}, P) \supset \text{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, \mathcal{A}, P),$$

is H -stochastic integrable with respect to the normal martingale N (i.e. belongs to S_2) if and only if F belongs to the space $L_a^2([0, T] \times \Omega)$.

The next theorem shows that the Itô stochastic integral with respect to the normal martingale N can be interpreted as an H -stochastic integral.

Theorem 3. *Let $F \in L_a^2([0, T] \times \Omega)$ and $\{A_F(t)\}_{t \in [0, T]}$ be the corresponding family of the operators $A_F(t)$ of multiplication by $F(t)$ in the space $L^2(\Omega, \mathcal{A}, P)$. Then*

$$\int_{[0, T]} A_F(t) dN_t = \int_{[0, T]} F(t) dN_t.$$

Proof. Taking into account Theorem 2, Lemma 1 and the definitions of the integrals

$$\int_{[0, T]} A_F(t) dN_t \quad \text{and} \quad \int_{[0, T]} F(t) dN_t,$$

it is sufficient to prove Theorem 3 for simple functions $F \in L_a^2([0, T] \times \Omega)$. But in this case Theorem 3 is obvious. \square

4. THE ITÔ INTEGRAL IN A FOCK SPACE AS AN H -STOCHASTIC INTEGRAL

Let us recall the definition of the Itô integral in a Fock space, see e.g. [2] for more details. We denote by \mathcal{F} the symmetric Fock space over the real separable Hilbert space $L^2([0, T]) := L^2([0, T], dt)$. By definition (see e.g. [5]),

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{F}_n n!,$$

where $\mathcal{F}_0 := \mathbb{C}$ and, for each $n \in \mathbb{N}$, $\mathcal{F}_n := (L_{\mathbb{C}}^2([0, T]))^{\widehat{\otimes} n}$ is an n -th symmetric tensor power $\widehat{\otimes}$ of the complex Hilbert space $L_{\mathbb{C}}^2([0, T])$. Thus, the Fock space \mathcal{F} is the complex Hilbert space of sequences $f = (f_n)_{n=0}^{\infty}$ such that $f_n \in \mathcal{F}_n$ and

$$\|f\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{F}_n}^2 n! < \infty.$$

We denote by $L^2([0, T]; \mathcal{F})$ the Hilbert space of all \mathcal{F} -valued functions

$$[0, T] \ni t \mapsto f(t) \in \mathcal{F}, \quad \|f\|_{L^2([0, T]; \mathcal{F})} := \left(\int_{[0, T]} \|f(t)\|_{\mathcal{F}}^2 dt \right)^{\frac{1}{2}} < \infty$$

with the corresponding scalar product. A function $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L^2([0, T]; \mathcal{F})$ is called *Itô integrable* if, for almost all $t \in [0, T]$,

$$f(t) = (f_0(t), \mathfrak{A}_{[0, t]} f_1(t), \dots, \mathfrak{A}_{[0, t]^n} f_n(t), \dots).$$

We denote by $L_a^2([0, T]; \mathcal{F})$ the set of all Itô integrable functions.

Let f belong to the space $L_{a,s}^2([0, T]; \mathcal{F})$ of all *simple Itô integrable functions*. That is, f belongs to $L_a^2([0, T]; \mathcal{F})$ and there exists a partition $0 = t_0 < t_1 < \dots < t_n = T$ of $[0, T]$ such that

$$f(t) = \sum_{k=0}^{n-1} f(k) \mathfrak{A}_{(t_k, t_{k+1}]}(t) \in \mathcal{F}$$

for almost all $t \in [0, T]$. The *Itô integral* $\mathbb{I}(f)$ of such a function f is defined by the formula

$$\mathbb{I}(f) := \sum_{k=0}^{n-1} f(k) \diamond (0, \mathfrak{A}_{(t_k, t_{k+1}]}(0, 0, \dots)) \in \mathcal{F},$$

where the symbol \diamond denotes the Wick product in the Fock space \mathcal{F} . Let us recall that for given $f = (f_n)_{n=0}^{\infty}$ and $g = (g_n)_{n=0}^{\infty}$ from \mathcal{F} the Wick product $f \diamond g$ is defined by

$$f \diamond g := \left(\sum_{m=0}^n f_m \widehat{\otimes} g_{n-m} \right)_{n=0}^{\infty},$$

provided the latter sequence belongs to the Fock space \mathcal{F} .

The Itô integral $\mathbb{I}(f)$ of a simple function $f \in L_{a,s}^2([0, T]; \mathcal{F})$ has the isometry property

$$\|\mathbb{I}(f)\|_{\mathcal{F}}^2 = \int_{[0, T]} \|f(t)\|_{\mathcal{F}}^2 dt,$$

see e.g. [2, 1]. Hence, extending the mapping

$$L_a^2([0, T]; \mathcal{F}) \supset L_{a,s}^2([0, T]; \mathcal{F}) \ni f \mapsto \mathbb{I}(f) \in \mathcal{F}$$

by continuity we obtain a definition of the Itô integral $\mathbb{I}(f)$ for each $f \in L_a^2([0, T]; \mathcal{F})$ (we keep the same notation \mathbb{I} for the extension).

Let us show that the Itô integral $\mathbb{I}(f)$ of $f \in L_a^2([0, T]; \mathcal{F})$ can be considered as an H -stochastic integral. To do this we set $\mathcal{H} := \mathcal{F}$ and consider in this space the resolution of identity

$$[0, T] \ni t \mapsto \mathcal{X}_t f := (f_0, \mathcal{X}_{[0, t]} f_1, \dots, \mathcal{X}_{[0, t]^n} f_n, \dots) \in \mathcal{L}(\mathcal{F}), \quad f = (f_n)_{n=0}^\infty \in \mathcal{F}.$$

Let $Z := (0, 1, 0, 0, \dots) \in \mathcal{F}$ and

$$[0, T] \ni t \mapsto Z_t := \mathcal{X}_t Z = (0, \mathcal{X}_{[0, t]} Z, 0, 0, \dots) \in \mathcal{F}$$

be the corresponding abstract martingale in the Fock space \mathcal{F} . Note that now

$$\mu([0, t]) := \|Z_t\|_{\mathcal{F}}^2 = \|\mathcal{X}_{[0, t]}\|_{L_{\mathcal{F}}^2([0, T])}^2 = t, \quad t \in [0, T],$$

i.e., μ is the Lebesgue measure on $\mathcal{B}([0, T])$.

We have the following analogues of Theorems 2 and 3.

Theorem 4. *A function $f \in L^2([0, T]; \mathcal{F})$ belongs to the space $L_a^2([0, T]; \mathcal{F})$ if and only if the corresponding operator-valued function $[0, T] \ni t \mapsto A_f(t)$ whose values are operators $A_f(t)$ of Wick multiplication by $f(t) \in \mathcal{F}$ in the Fock space \mathcal{F} ,*

$$\mathcal{F} \supset \text{Dom}(A_f(t)) \ni g \mapsto A_f(t)g := f(t) \diamond g \in \mathcal{F},$$

belongs to the space S_2 .

Theorem 5. *Let $f \in L_a^2([0, T]; \mathcal{F})$ and $\{A_f(t)\}_{t \in [0, T]}$ be the corresponding family of the operators $A_f(t)$ of Wick multiplication by $f(t) \in \mathcal{F}$ in the Fock space \mathcal{F} . Then*

$$\mathbb{I}(f) = \int_{[0, T]} A_f(t) dZ_t.$$

Taking into account Theorem 5, in what follows we will denote the Itô integral $\mathbb{I}(f)$ of $f \in L_a^2([0, T]; \mathcal{F})$ by $\int_{[0, T]} f(t) dZ_t$. Note that this integral can be expressed in terms of the Fock space \mathcal{F} . Namely, for any $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L_a^2([0, T]; \mathcal{F})$, we have

$$(8) \quad \int_{[0, T]} f(t) dZ_t = (0, \hat{f}_1, \dots, \hat{f}_n, \dots) \in \mathcal{F},$$

where, for each $n \in \mathbb{N}$ and almost all $(t_1, \dots, t_n) \in [0, T]^n$,

$$\hat{f}_n(t_1, \dots, t_n) := \frac{1}{n} \sum_{k=1}^n f_{n-1}(t_k; t_1, \dots, \check{t}_k, \dots, t_n),$$

i.e., \hat{f}_n is the symmetrization of $f_{n-1}(t; t_1, \dots, t_{n-1})$ with respect to n variables.

5. A CONNECTION BETWEEN THE CLASSICAL ITÔ INTEGRAL AND THE ITÔ INTEGRAL IN THE FOCK SPACE

As before, let (Ω, \mathcal{A}, P) be a complete probability space with a right continuous filtration $\{\mathcal{A}_t\}_{t \in [0, T]}$, \mathcal{A}_0 be the trivial σ -algebra containing all P -null sets of \mathcal{A} and $\mathcal{A} = \mathcal{A}_T$.

Let $N = \{N_t\}_{t \in [0, T]}$ be a normal martingale on (Ω, \mathcal{A}, P) with respect to $\{\mathcal{A}_t\}_{t \in [0, T]}$, $N_0 = 0$. It is known that the mapping

$$\mathcal{F} \ni f = (f_n)_{n=0}^\infty \mapsto If := \sum_{n=0}^\infty I_n(f_n) \in L^2(\Omega, \mathcal{A}, P)$$

is well-defined and isometric. Here $I_0(f_0) := f_0$ and, for each $n \in \mathbb{N}$,

$$I_n(f_n) := n! \int_0^T \int_0^{t_n} \cdots \left(\int_0^{t_2} f_n(t_1, \dots, t_n) dN_{t_1} \right) \dots dN_{t_{n-1}} dN_{t_n}$$

is an n -iterated Itô integral with respect to N . We suppose that the normal martingale N has the *chaotic representation property* (CRP). In other words, we assume that the mapping $I : \mathcal{F} \rightarrow L^2(\Omega, \mathcal{A}, P)$ is a unitary. Note that

$$N_t = IZ_t \in L^2(\Omega, \mathcal{A}, P), \quad t \in [0, T],$$

i.e., N is the I -image of the abstract martingale $[0, T] \ni t \mapsto Z_t = (0, \varkappa_{[0, t]}, 0, 0, \dots) \in \mathcal{F}$.

The Brownian motion, the compensated Poisson process and some Azéma martingales are examples of normal martingales which possess the CRP, see e.g. [10, 11].

We note that the spaces $L^2([0, T] \times \Omega)$ and $L^2([0, T]; \mathcal{F})$ can be understood as the tensor products $L^2([0, T]) \otimes L^2(\Omega, \mathcal{A}, P)$ and $L^2([0, T]) \otimes \mathcal{F}$, respectively. Therefore,

$$1 \otimes I : L^2([0, T]; \mathcal{F}) \rightarrow L^2([0, T] \times \Omega)$$

is a unitary operator.

The next result gives a relationship between the classical Itô integral with respect to the normal martingale with CRP and the Itô integral in the Fock space \mathcal{F} .

Theorem 6. *We have*

$$L_a^2([0, T] \times \Omega) = (1 \otimes I)L_a^2([0, T]; \mathcal{F})$$

and, for arbitrary $f \in L_a^2([0, T]; \mathcal{F})$,

$$I\left(\int_{[0, T]} f(t) dZ_t\right) = \int_{[0, T]} If(t) dN_t.$$

Since N has CRP, for any $F \in L_a^2([0, T] \times \Omega)$ there exists a uniquely defined vector $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L_a^2([0, T]; \mathcal{F})$ such that

$$F(t) = If(t) = \sum_{n=0}^\infty I_n(f_n(t))$$

for almost all $t \in [0, T]$. Hence, using Theorem 6 and equality (8) we obtain

$$\int_{[0, T]} F(t) dN_t = I\left(\int_{[0, T]} f(t) dZ_t\right) = \sum_{n=1}^\infty I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P).$$

It should be noticed that the right hand side of the latter equality was used by Hitsuda [9] and Skorohod [14] to define an extension of the Itô integral. Namely, a function

$$F(\cdot) = \sum_{n=0}^\infty I_n(f_n(\cdot)) \in L^2([0, T] \times \Omega)$$

is Hitsuda-Skorohod integrable if and only if

$$\sum_{n=1}^\infty I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P) \quad \text{or, equivalently,} \quad \sum_{n=1}^\infty \|\hat{f}_n\|_{\mathcal{F}_n}^2 n! < \infty.$$

The corresponding *Hitsuda-Skorohod integral* $\mathbb{I}_{\text{HS}}(F)$ of F is defined by the formula

$$\mathbb{I}_{\text{HS}}(F) := \sum_{n=1}^\infty I_n(\hat{f}_n).$$

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